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Forms and cohomology of supermanifolds under review

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Abstract

It is shown that the particular generalization of the de Rham complex to smooth Berezin–Leites–Kostant supermanifolds proposed in Voronov and Zorich (1986, 1988) and Voronov (1991) is not self-consistent. Some additional conditions are needed in order to make it work. It is proved that when these conditions are taken into account the cohomology of the resultant complex is isomorphic to the usual de Rham cohomology of the underlying manifold.

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1. Introduction

Supermanifold theory has its roots in the fact that quantum field theory describes fermions at the classical level by anticommuting fields. The pioneering work of Berezin in the early 1970s lead to the conclusion that both, commuting and anticommuting variables, should appear on equal footing in supermanifolds. Several theories of supermanifolds were developed, and theoretical physicists soon requested the appropriate – and presumably new – mathematical tools to understand the geometry and the topology of supermanifolds in order to provide solid foundations for their physical content. Thus, since the origins of the subject, several analogs of classical geometrical and topological structures have been generalized to the various theories of supermanifolds (Lie superalgebras and their representations; deformations of complex supermanifolds; Berezinian integration; Serre duality; etc., just

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to mention a few developments). This article, in particular, deals with the generalization proposed in [30–32] of the de Rham cohomolgy to the category of smooth Berezin–Leites–Kostant supermanifolds. We actually show that the definition of such de Rham cohomology is incomplete as given in those references, but that there is a unique way of turning it into a consistent one. After supplying the definition with the appropriate conditions, we show that the resulting de Rham cohomology is isomorphic to the usual de Rham cohomology of the underlying manifold. Thus, in spite of what had been conjectured in [30–32], this cohomology is not sensitive to the superstructure and cannot be used as a tool to encode the information given by the fermionic (anticommuting) variables.

It is worth saying a few words about the background of this problem in order to better place our results in the existing literature. First of all, as we have already mentioned it, the implemention of commuting and anticommuting fields into a single analytic scheme resulted into the development of several types of supermanifolds, of which, two main streams are the most followed: The theory of G-supermanifolds, and the theory of Berezin–Leites–Kostant supermanifolds (also called graded manifolds). Amongst the great number of articles dealing with these two approaches we refer the reader to [11] (specially Sections III and V) and [26] for a comparison between them from the physical point of view. From the mathematical point of view, on the other hand, G-supermanifolds are the closest to DeWitt's [12] and Rogers' [23] original theory of supermanifolds; they furthermore satisfy Rothstein's axiomatics [24]. Moreover, the category of G-supermanifolds contains as full subcategories several types of supermanifolds introduced earlier as separate theories (e.g., DeWitt, and H^∞, among others). An in-depth development of the category of G-supermanifolds can be found in the monograph [3]. On the other hand, Berezin–Leites–Kostant supermanifolds were independently introduced by Berezin and Leites [6,7,17], and by Kostant [16] (see also the book by Manin [19] for a thorough introduction to this theory). Actually, it has been established in [3] that there is a one-to-one correspondence between isomorphism classes of DeWitt supermanifolds and isomorphism classes of Berezin–Leites–Kostant supermanifolds, although the correspondence is not a functorial one.

Let us now recall the main properties of the cohomology theories on supermanifolds developed so far. First of all, a general $\mathbb{Z}$-graded de Rham-like cohomology theory was constructed in [2] for G-supermanifolds (see also [3,10]). This theory has been proved in [2] to be sensitive to the 'super' structure except when the given G-supermanifold is of DeWitt type; in the latter case the $\mathbb{Z}$-graded de Rham cohomology is determined by the de Rham cohomology of the body (see [2,3,10,22]). For Berezin–Leites–Kostant supermanifolds on the other hand, there are at least six types of cohomology theories in the literature of which the author is aware of: (1) The $\mathbb{Z}$-graded de Rham-like cohomology of even and odd forms (see [19]). In this case a Poincaré lemma can be proved and the corresponding cohomology becomes isomorphic to the de Rham cohomology of the underlying smooth manifold (see [16,19]). (2) The $\mathbb{Z}$-graded cohomology of integral forms (see [8]). It is proved in [20] that the complex of integral forms is quasi-isomorphic to the ground field; therefore, this cohomology does not give any useful information about the supermanifold. (3) The ungraded cohomology of pseudodifferential forms developed in [9], which does not compare to anything in the graded context. (4) The $\mathbb{Z}^2$-graded de Rham-like cohomology introduced in [1], further developed in [30,31], and recasted in the monograph [32]. This theory emerges as a response to the problem of developing a non-trivially $\mathbb{Z}^2$-graded cohomology, not determined by the de Rham cohomology of the underlying smooth manifold, and thus giving an answer to problem (8) in the list of [18] (see also [21,29]). The review of this particular theory is the main concern of this paper: It is proved here that this theory does not produce a suitable cohomology in the sense of problem (8) of [18]. (5) The $\mathbb{Z}^2$-graded de Rham-like cohomology developed in [28]. A $\mathbb{Z}^2$-graded de Rham cohomology functor – not determined by the usual de Rham cohomology of the underlying manifold – was introduced there. It was shown that it does provide an answer to problem (8) of [18], and furthermore, that it satisfies the requirements in [21,29] for a suitable cohomology on Berezin–Leites–Kostant supermanifolds sensitive to the 'super' structure. (6) Finally, one may also consider the cohomology of the (isomorphism class of the) G-supermanifold of DeWitt type that corresponds to a given (isomorphism class of) Berezin–Leites–Kostant supermanifold under the correspondence layed out in [3]. This, however, cannot be compared to the problem we are dealing with in this paper. The first reason is that, whereas the first five theories we have recalled above are functorially defined within the category of Berezin–Leites–Kostant supermanifolds, the correspondence made in [3] is not functorial. Secondly, the 'DeWitt cohomology' obtained this way for Berezin–Leites–Kostant supermanifolds does not provide non-trivially $\mathbb{Z}^2$-graded groups, and furthermore, it is completely determined by the cohomology of the body.

Apart from its mathematical interest, cohomology of supermanifolds has various physical applications; mainly related with quantization of supergauge and superstring theories (e.g., see [22]). It was pointed out in [33], for instance, the necessity of investigating the geometrical nature of the Batalin–Vilkovisky superspace formalism in the quantization of gauge field theories. It was argued in [14] that the reason for introducing antifields in the Batalin–Vilkovisky formalism is connected with the integration theory on supermanifolds; concretely, with the integration of pseudodifferential forms (as in [9]) and of the $\mathbb{Z}^2$-graded forms of [1,30–32]. The relationship between Batalin–Vilkovisky formalism and these $\mathbb{Z}^2$-forms have been studied in [15] by making explicit use of the results of [30–32]. On the other hand, the multi-loop superstring amplitude computed in [4,5] is based on the integration theory of the $\mathbb{Z}^2$-graded forms of [30–32]. Moreover, the integration and cohomology of the $\mathbb{Z}^2$-graded forms of [30–32] are used in [5] to provide a new geometrical approach to superstrings. It is at the light of these applications and developments that our findings become relevant, and might shed some light in the near future onto the problem of elucidating some relationships between integration, de Rham cohomology of $\mathbb{Z}^2$-graded forms, and its physical applications.

In this paper supermanifolds are from now on Berezin–Leites–Kostant supermanifolds. It is organized as follows: We first recall in Section 2 the definitions of lagrangians, forms, and the operator d as given in [30–32]. We show in Section 3 that d^2 – as defined there – does not vanish, and we find out what are the precise conditions that must be added to the definition of forms in order that it does. The computation of the sheaves (resp. the cohomology) of the resultant complex is done in Theorems 11 and 15 (resp. Theorem 16) of Section 4.

2. Lagrangians and forms on supermanifolds

We shall take the definitions from [1,30–32]. Since Refs. [1,30,31] have been recapitulated in the book [32], we shall only refer to the later from now on. Our notation adheres essentially to that of [19,32] with the following convention: lower-case latin (resp. greek) indices are even (resp. odd), while upper-case latin indices may be even or odd. For instance, if $\{x^A\}$ is a system of local coordinates, the parity $\widetilde{x^A}$ of x^A is

$$\widetilde{x^A} = \widetilde{A} = \begin{cases} 0 & \text{if } A \text{ is lower-case latin,} \\ 1 & \text{if } A \text{ is lower-case greek.} \end{cases}$$

For any pair of non-negative integers r, s, the $r|s$-tangent superbundle to M is the fibered supermanifold $\pi_{r|s} : T^{r|s}M \to M$, described in terms of local coordinates as follows: any system $\{x^A\}$ of local coordinates on $U \subset M$, $\dim(M) = (m_0, m_1)$ gives rise to a system $\{x_F^A\}$ of local coordinates on the fibers on ${\pi_{r|s}}^{-1}(U)$, with parities $\widetilde{x_F^A} = \widetilde{A} + \widetilde{F}$, where the first group of capital letters $A, B, \dots$ (resp. the second one $F, G, \dots$) are used to denote generic superindices in $\{1, \dots, m_0+m_1\}$, $\dim(M) = (m_0, m_1)$ (resp. subindices in $\mathbb{N}\setminus\{0\}$) whose parities are

$$\widetilde{A} = \begin{cases} 0 & \text{if } 1 \le A \le m_0, \\ 1 & \text{if } m_0 + 1 \le A \le m_0 + m_1, \end{cases} \quad \text{resp.} \quad \widetilde{F} = \begin{cases} 0 & \text{if } 1 \le F \le r, \\ 1 & \text{if } r + 1 \le F \le r + s. \end{cases}$$

The relationship between any two such systems on the fibers is $y_F^A = \sum_B x_F^B \partial y^A / \partial x^B$. These constructions are functorial: any morphism of supermanifolds $f: M \to N$ defines a morphism of fibered supermanifolds $f_{r|s}: T^{r|s}M \to T^{r|s}N$ which locally acts on the fibers as $f_{r|s}^*(y_F^A) = \sum_B x_F^B \partial f^*(y^A)/\partial x^B$. With the notations of [19], $T^{r|s}M = T^r M \oplus \Pi\, T^s M$. We shall refer to [25,27] for a more intrinsic description of the tangent superbundles and their significance in the theory of supermanifolds.

Finally, to simplify the notation in what follows, we introduce the following local differential operators:

$$\partial_{GF}^{BA} = \begin{cases} \dfrac{\partial^2}{\partial x_G^B \partial x_F^A} + (-1)^{\widetilde{F}\widetilde{G} + \widetilde{A}(\widetilde{F}+\widetilde{G})} \dfrac{\partial^2}{\partial x_F^B \partial x_G^A} & \text{if } F \ne G, \\ \dfrac{\partial^2}{\partial x_G^B \partial x_F^A} & \text{if } F = G. \end{cases}$$

We may now quote the following three definitions:

Definition 1 [32, p. 57]. $W_{r|s}(M)$ is defined to be the subsupermanifold of $T^{r|s}M$ locally singled out by the equations $\operatorname{rank}(x_\mu^\alpha) = s$. The sheaf of $r|s$-lagrangian on M is the sheaf of superfunctions on $W_{r|s}(M)$.

To stress that a $r|s$-lagrangian L locally depends on x^A and on the matrix of coordinates x_F^A, we write $L = L(x^A, x_F^A)$ or $L = L(x^A, x_1, \dots, x_{r+s})$ where x_F, $F = 1, \dots, r+s$ denotes the vector of coordinates x_F^A, $A = 1, \dots, m_0 + m_1$, $\dim(M) = (m_0, m_1)$.

Definition 2 [32, pp. 56–58]. A $r|s$-lagrangian L on M is said to be a $r|s$-form on M if
(1) for any matrix $g \in \mathrm{GL}(r|s)$,

$$L\left(x^A, \sum_B x^A_G g^G_F\right) = \mathrm{Ber}(g) L(x^A, x^A_F),$$

where $\mathrm{Ber}(g)$ is the berezinian of the matrix g. We refer ourselves to this equation as the *berezinian condition*.
(2) $\partial^{BA}_{GF}(L) = 0,\ \forall A, B, F \neq G$.

These forms are functorial only with respect to the morphisms of supermanifolds $f : M \to N$ which satisfy $f_{r|s}(W_{r|s}(M)) \subset W_{r|s}(N)$. To ensure this inclusion one must be restricted to the strict subcategory of supermanifolds with morphisms which are immersions with respect to the odd variables. These morphisms are called proper morphisms [32, p. 57].

Definition 3 [32, p. 58]. The differential of a $r|s$-form L is the $(r+1|s)$-form

$$\mathrm{d}L = (-1)^r \sum_A x^A_{r+1} \left(\frac{\partial}{\partial x^A} - \sum_{B,F} (-1)^{\widetilde{A}\widetilde{F}} x^B_F \frac{\partial^2}{\partial x^B \partial x^A_F} \right) L.$$

Forms annihlated by d are called closed, and forms like $\mathrm{d}L$ are called exact. It is argued in [32, p. 61] that $\mathrm{d}^2 L = 0$ for any form L, and therefore one can consider the quotient groups of closed forms modulo the exact ones. These are the cohomology groups proposed in [32] as a generalization to supermanifolds of the usual de Rham cohomology groups on manifolds.

We will see in Section 3 that d^2 does not necessarily vanish on $r|s$-forms. So there is neither complex of $r|s$-forms (as they are defined in Definition 2) nor cohomology of them (but see Definition 5).

3. On the definition of $r|s$-forms

In this section we prove that in order to ensure that $\mathrm{d}^2 L = 0$ for any $r|s$-form L, L must satisfy not only Eqs. (1) and (2) in Definition 2 (as it is said in [32, pp. 57–58]), but moreover Eq. (2) in Theorem 4.

First of all let us recall the following result:

Theorem 4. *Let M be a supermanifold and $r, s \in \mathbb{N}$.*
(1) *There exists a morphism of sheaves of commutative groups*

$$D_{r|s} : (\pi_{r|s})_* \mathcal{O}_{T^{r|s}M} \to (\pi_{r+1|s})_* \mathcal{O}_{T^{r+1|s}M}$$

locally written as $(-1)^r \sum_A x^A_{r+1} \left(\frac{\partial}{\partial x^A} - \sum_{B,F} (-1)^{\widetilde{A}\widetilde{F}} x^B_F \frac{\partial^2}{\partial x^B \partial x^A_F} \right)$.

(2) *The sections* L *of* $(\pi_{r|s})_*\mathcal{O}_{T^{r|s}M}$ *which are solutions of the equations*

$$\partial_{GF}^{BA}(L)=0,\quad \forall A, B, F, G$$

define a subsheaf $\mathcal{F}_M^{r|s}$ *of* $\mathcal{O}_M$*-modules of* $(\pi_{r|s})_*\mathcal{O}_{T^{r|s}M}$.

(3) $D_{r+1|s}\circ D_{r|s}=0$ *on* $\mathcal{F}_M^{r|s}$.

(4) $D_{r|s}(\mathcal{F}_M^{r|s})\subset \mathcal{F}_M^{r+1|s}$.

Proof. Part (1), resp. (2), is part (1), resp. (3), in Theorem 2.2.1 in [28] and part (3) is Proposition 2.2.1 in [28] (the notational correspondences are: superindices here are subindices in [28] and vice versa, and A, B, F, G here are G, F, B, A, resp. in [28].) □

Since the proof of part (3) has special interest for us we shall sketch it here:

$$D^2=\sum_{F\neq G}\pm x_{r+1}^{B}x_G^{C}\frac{\partial}{\partial x^{C}}x_r^{B'}x_F^{C'}\frac{\partial}{\partial x^{C'}}\partial_{GF}^{BA}+\sum_{F}\pm x_{r+1}^{B}x_F^{C}\frac{\partial}{\partial x^{C}}x_r^{B'}x_F^{C'}\frac{\partial}{\partial x^{C'}}\partial_{FF}^{BA}.$$

Therefore, to obtain a subsheaf $\mathcal{F}_M^{r|s}$ of $\mathcal{O}_M$-modules of $(\pi_{r|s})_*\mathcal{O}_{T^{r|s}M}$ on which D^2 vanishes one has to impose $\partial_{GF}^{BA}(L)=0$, $F\neq G$ and moreover $\partial_{FF}^{BA}(L)=0$. This suggests that equations $\partial_{FF}^{BA}=0$ must be added to conditions $\partial_{GF}^{BA}=0$, $F\neq G$ in part (2) of Definition 2. Beside, let us now recall from [32, pp. 56–58] the variational considerations on which Definition 2 of $r|s$-forms are based. Let I and M be supermanifolds of dimension (r,s) and (m,n), respectively, and $f: I\to M$ be a proper morphism. From part (1) in Definition 2 the pullback $f_{r|s}^*(L)$ of a $r|s$-form L on M is a section of the berezinian sheaf of I. Assuming that the underlying manifold I is compact and oriented, the action $S(f,L)=\int_I f_{r|s}^*(L)$ is well defined, where $\int_I$ is the berezinian integral. By computing the variation of this action one obtains

$$\frac{\delta S}{\delta x^A}=\frac{\partial L}{\partial x^A}-\sum_F(-1)^{\widetilde{AF}}\frac{\partial}{\partial t^F}\frac{\partial L}{\partial x_F^A}=\frac{\delta L}{\delta x^A}-\sum_{F,G,B}(-1)^{\widetilde{AF}}\frac{\partial x_G^B}{\partial t^F}\frac{\partial^2 L}{\partial x_G^B\partial x_F^A},$$

where

$$\frac{\delta L}{\delta x^A}=\frac{\partial L}{\partial x^A}-\sum_{F,B}(-1)^{\widetilde{AF}}\frac{\partial x_B}{\partial t_F}\frac{\partial^2 L}{\partial x^B\partial x_F^A}.$$

This is exactly the same as what is done in [32, p. 57]. The mistake in the computantions of [32] is in the expansion of $\delta S/\delta x^A-\delta L/\delta x^A$, which is

$$-\sum_{B,F\neq G}(-1)^{\widetilde{AF}}\frac{\partial x_G^B}{\partial t^F}\partial_{GF}^{BA}(L)-\sum_{B,F}(-1)^{\widetilde{AF}}\frac{\partial x_F^B}{\partial t^F}\partial_{FF}^{BA}(L)$$

instead of the expression given in [32, p. 57] where $\partial_{FF}^{BA}(L)$ does not appear. So, these variational considerations as well as Theorem 4 suggest that conditions $\partial_{FF}^{BA}=0$ are indispensable in the definition of the searched complexes. In fact, it is easy to verify that

$L = x^1x^2/x_1^1$ is a local 0|1-form on $\mathbb{R}^{0|2}$ such that $d^2L \neq 0$. As a consequence, to make the definition of the complexes self-consistent Definition 2 has to be rewritten as:

Definition 5. A $r|s$-lagrangian L on M is said to be a $r|s$-form on M if
(1) for any matrix $g \in \mathrm{GL}(r|s)$,

$$L\left(x^A, \sum_B x_G^A g_F^G\right) = \mathrm{Ber}(g) L(x^A, x_F^A),$$

where $\mathrm{Ber}(g)$ is the berezinian of the matrix g. We refer ourselves to this equation as the *berezinian condition*.
(2) $\partial_{GF}^{BA}(L) = 0,\ \forall A, B, F, G.$

Remark 6. Observe that for purely even supermanifolds, that is manifolds, Definitions 2 and 5 are equivalent, but it is not the case for general supermanifolds.

From now on $r|s$-forms are always understood in the sense of Definition 5.

4. Description of $r|s$-forms

We use the following version of the Taylor expansion for local functions f on supermanifolds of dimension (m_0, m_1) with respect to a system of coordinates x^A where $\widetilde{x^a} = 0$ for $1 \leq a \leq m_0$ and $\widetilde{x^\alpha} = 1$ for $m_0 \leq \alpha \leq m_0 + m_1$:

$$f(x^A) = \sum_{\substack{0 \leq k \leq m_1 \\ \alpha_i < \alpha_{i+1}}} x^{\alpha_1} \cdots x^{\alpha_k} f_{\alpha_1 \cdots \alpha_k}(x^a) = \sum_{|\underline{\alpha}| \geq 0} x^{\underline{\alpha}} f_{\underline{\alpha}}(x^a),$$

where $\underline{\alpha}$ stands for a multi-index $\alpha_1 < \cdots < \alpha_k$, $\alpha_i \in \{m_0 + 1, \ldots, m_0 + m_1\}$, $k = |\underline{\alpha}| = \sum_i \alpha_i \geq 0$ (recall the assumptions on notations made in Section 2). For the sake of simplification, define $\Delta_{\underline{a}\,\underline{\alpha}}$ to be

$$(-1)^{|\underline{\alpha}|(|\underline{\alpha}|-1)/2} \frac{\partial^{|\underline{a}|+|\underline{\alpha}|}}{\partial x^{a_1} \cdots \partial x^{a_k} \partial x^{\alpha_1} \cdots \partial x^{\alpha_k}}.$$

Then

$$f(x^A) = \sum_{\substack{0 \leq |a| \leq p \\ 0 \leq |\alpha|}} x^{\underline{a}} x^{\underline{\alpha}} \Delta_{\underline{a}\,\underline{\alpha}} f(0) + \sum_{\substack{|a| = p+1 \\ 0 \leq |\alpha|}} x^{\underline{a}} x^{\underline{\alpha}} \Delta_{\underline{a}\,\underline{\alpha}} f(\xi),$$

where $\underline{a}$ is a multi-index $a_1, \ldots, a_k$, $a_i \in \{1, \ldots, m_0\}$, $|\underline{a}| = \sum_i a_i$. Therefore we have:

Lemma 7. *If* $\partial^2 f/\partial x^A \partial x^B = 0\ \forall A, B$, *then* $f(x^A) = \sum_{0 \leq |\underline{\alpha}| + r \leq 1} x^{\underline{\alpha}} P_{r,\underline{\alpha}}(x^a)$ *where* $P_{r,\underline{\alpha}} \in \mathbb{R}[x^1, \ldots, x^{m_0}]$ *is a polynomial of degree* r.

From the condition $\partial_{FF}^{BA}(L) = 0$ and Lemma 7 we obtain for $r + s = 1$:

$$L(x^A, x_1) = L(x^A, 0) + \sum_B x_1^B \frac{\partial L}{\partial x_1^B}(x^A, 0),$$

for $r+s=2$:

$$L(x^A, x_1, x_2) = L(x^A, 0, 0) + \sum_B x_1^B \frac{\partial L}{\partial x_1^B}(x^A, 0, 0) + \sum_{C',C} x_2^{C'} \left(\frac{\partial L}{\partial x_2^{C'}}(x^A, 0, 0) + x_1^C \frac{\partial^2 L}{\partial x_1^C \partial x_2^{C'}}(x^A, 0, 0) \right)$$

and inductively one has the following lemma.

Lemma 8. *If L is a $r|s$-lagrangian such that $\partial_{FF}^{BA}(L) = 0\ \forall A, B, F$ then, locally*

$$L = \sum_{\substack{0 \le k \le r+s \\ 1 \le A_i \le m_0 + m_1 \\ 1 \le F_i < F_{i+1} \le r+s}} x_{F_1}^{A_1} \cdots x_{F_k}^{A_k} L_{A_1,\dots,A_k}^{F_1,\dots,F_k}(x^A),$$

where the number of even, resp. odd, indices F_i's is $\le r$, resp. $\le s$.

Moreover, we have:

Proposition 9. *If L is a $r|s$-lagrangian which is a $r|s$-form, then the local coefficients of $L_{A_1,\dots,A_k}^{F_1,\dots,F_k}$ in Lemma 8 satisfy*

$$L_{\cdots A_j \cdots A_{j'} \cdots}^{\cdots F_j \cdots F_{j'} \cdots} + (-1)^{\widetilde{A}_j \widetilde{A}_{j'} + (\widetilde{A}_j + \widetilde{A}_{j'})\left(\widetilde{F}_{j'} + \sum_{j<i<j'} \widetilde{A}_i + \widetilde{F}_i\right)} L_{\cdots A_{j'} \cdots A_j \cdots}^{\cdots F_j \cdots F_{j'} \cdots} = 0.$$

Proof. These equations are straightforwardly obtained by imposing $\partial_{GF}^{BA}(L) = 0$, $\forall A, B$, $F \ne G$ to the expressions of Lemma 8. □

From Lemma 8 any $r|s$-form L can be written as a sum $\sum_{r' \le r, s' \le s} L_{r'|s'}$ where $L_{r'|s'}$ is a $r'|s'$-form with exactly r' even indices F_i's and exactly s' odd indices F_i's. This decomposition is not only local. It is global due to the homogeneity of the change of coordinates of the fibers of the tangent superbundles (recall Section 1). Forms of type $L_{r|s}$ will be called homogeneous $r|s$-forms.

Definition 10. We denote the sheaf of $\mathcal{O}_M$-modules of homogeneous $r|s$-forms on the supermanifold M as $\Omega_M^{r|s}$. $\Omega_{M_{\text{red}}}^p$ denotes the sheaf of $\mathcal{O}_{M_{\text{red}}}$-modules of usual p-forms on the underlying manifold M_{red} of the supermanifold M.

To have a complete description of $\Omega_M^{r|s}$ it remains to impose the berezinian condition. Before doing it, notice that it seems to be (and Theorem 11 below proves that actually, there is) an incompatibility between the conditions $\partial_{GF}^{BA} = 0\ \forall A, B, F, G$ and the berezinian condition. Roughly speaking this incompatibility emerges from the fact that Proposition 9 tells that forms are tensorial objects on the coordinates x_F^A, but the berezinian condition is not tensorial due to the presence of a denominator in the definition of the berezinian.

Theorem 11. *For any supermanifold M it holds $\Omega_M^{r|s} = 0\ \forall s > 0,\ \forall r$.*

Proof. Let L be a homogeneous $r|s$-form on the supermanifold M, $\dim(M) = (m_0, m_1)$. Let us consider the following cases:

Case $m_1 = 0$: Definition 1 forces $\Omega_M^{r|s}$ to be zero for any $s > 0$.

Case $m_1 > 0$: Consider a homogeneous $r|s$-form L. Thanks to Lemma 8 it can be locally written as $L = \sum_{A,F} x_{F_1}^{A_1} \cdots x_{F_{r+s}}^{A_{r+s}} L_{A_1 \cdots A_{r+s}}^{F_1 \cdots F_{r+s}}$. For any $\alpha \in \mathbb{K}^*$ let $g_\alpha \in \mathrm{GL}(r|s)$ be

$$(g_\alpha)_A^B = \begin{cases} \delta_A^B & \text{if } 1 \le A \le r, \\ \alpha\delta_A^B & \text{if } r+1 \le A \le r+s. \end{cases}$$

The berezinian condition is $\alpha^s L = (1/\alpha^s)L$. This forces L to be identically zero. So $\Omega_M^{r|s} = 0\, \forall s > 0$. □

The next theorem identifies $\Omega_M^{r|0}$. For that let us recall from [13, pp. 348–350] the following two definitions:

Definition 12. The r-fold antisymmetric tensors of an $\mathcal{O}_M$-module $\mathcal{V}$ is the subspace of $\bigotimes^r \mathcal{V}$ composed of tensor changing sign under all $P_{j,j'}$, where

$$\begin{aligned} &P_{jj'}(x_{F_1}^{A_1} \otimes \cdots \otimes x_{F_j}^{A_j} \otimes \cdots \otimes x_{F_{j'}}^{A_{j'}} \otimes \cdots \otimes x_{F_r}^{A_r}) \\ &\quad = (-1)^{(j,j')\underline{A}} x_{F_1}^{A_1} \otimes \cdots \otimes x_{F_j}^{A_{j'}} \otimes \cdots \otimes x_{F_{j'}}^{A_j} \otimes \cdots \otimes x_{F_r}^{A_r}), \end{aligned}$$

$(j, j')\underline{A} = \widetilde{A_j}\widetilde{A_{j'}} + (\widetilde{A_j} + \widetilde{A_{j'}})(\widetilde{A_{j+1} + \cdots + A_{j'-1}})$. Such a tensor is a linear combination of elements of the form $\sum_{\sigma \in S_r} (-1)^{\mathrm{sgn}(\sigma) + \sigma\underline{A}} x_{F_1}^{A_{\sigma_1}} \otimes \cdots \otimes x_{F_r}^{A_{\sigma_r}}$ where $\mathrm{sgn}(\sigma)$ is the number of transpositions mod 2 of the permutation σ, and $\sigma\underline{A}$ is the representation generated by $(j, j')\underline{A}$. They constitute an $\mathcal{O}_M$-module noted as $\Lambda_{\mathcal{V}}^r$.

Definition 13. The sheaf of even r-forms on the supermanifold M is the $\mathcal{O}_M$-module $\Omega_M^r = \Lambda_{(\mathrm{Der}\,\mathcal{O}_M)^*}^r$. There is a well-defined exterior differential $\mathbf{d} : \Omega_M^r \to \Omega_M^{r+1}$ whose square is zero. (This complex has also been considered in [16, pp. 244–250; 19, pp. 168–170].)

Remark 14. If $\{\mathbf{d}x^A\}$ is the dual of the local derivations $\{\partial/\partial x^A\}$, then $\mathbf{d}x^A\, \mathbf{d}x^B = -(-1)^{\tilde{A}\tilde{B}}\, \mathbf{d}x^B \mathbf{d}x^A$ (see [13]).

Theorem 15. *The complex* $(\Omega_M^{(\bullet|0)}, \mathrm{d})$ *is naturally isomorphic to the complex* $(\Omega_M^\bullet, \mathbf{d})$.

Proof. By Lemma 8 let us locally write any homogeneous $(r|0)$-form L as

$$L = \sum_A x_1^{A_1} \cdots x_r^{A_r} L_{A_1,\ldots,A_k}^{1 \cdots r}.$$

By Proposition 9 $L_{\cdots A_j \cdots A_{j'} \cdots}^{\cdots j \cdots j' \cdots} = (-1)^{\mathrm{sgn}(j,j') + (j,j')\underline{A}} L_{\cdots A_{j'} \cdots A_j \cdots}^{\cdots j \cdots j' \cdots}$. Then

$$L = \sum_{A_i \le A_{i+1}} \sum_{\sigma \in S_r} (-1)^{\mathrm{sgn}(\sigma) + \sigma\underline{A}} x_1^{A_{\sigma_1}} \cdots x_r^{A_{\sigma_r}} L_{A_1,\ldots,A_k}^{1 \cdots r}.$$

Now, the natural local linear morphism

$$\sum_{\sigma \in \mathcal{S}_r} (-1)^{\mathrm{sgn}(\sigma)+\sigma \underline{A}} x_1^{A_{\sigma_1}} \cdots x_r^{A_{\sigma r}} \longrightarrow \mathbf{d}x^{A_1} \cdots \mathbf{d}x^{A_r}$$

commutes with d and **d** and induces a well-defined global isomorphism between $(\Omega_M^{(\bullet|0)}, \mathrm{d})$ and $(\Omega_M^{\bullet}, \mathbf{d})$. □

It is well known that Ω_M^r is a locally free sheaf over $\mathcal{O}_M$ and how each system of local coordinates on M induces a basis on it [13,16,19]. This gives a simple explicit description of the finite-dimensional module of $r|s$-forms.

Notice that all the previous constructions and results are valid for supermanifolds over the ground field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. That is why we write $\mathcal{O}_M$ to denote the sheaf of superfunctions instead of C_M^∞ which is the usual notation for the specific case $\mathbb{K} = \mathbb{R}$.

Finally let us show that the cohomology of the complex of $\bullet|\bullet$-forms of an $\mathbb{R}$-differentiable supermanifold M is isomorphic to the usual de Rham cohomology of its underlying manifold M_{red}.

Theorem 16. *For any $\mathbb{R}$-differenciable supermanifold M, $H^\bullet(\Omega_M^{\bullet|0}, \mathrm{d}) \simeq H^\bullet(M_{\mathrm{red}})$.*

Proof. By Theorem 15 $H^\bullet(\Omega_M^{\bullet|0}, \mathrm{d}) \simeq H^\bullet(M)$ and it is well known (see [16] or [19]) that this later group is isomorphic to $H^\bullet(M_{\mathrm{red}})$. □

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